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Localization Phenomenon in Gaps of the Spectrum of Random Lattice Operators

Alexander Figotin[1] and Abel Klein[2]

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We consider a class of random lattice operators including Schrödinger operators of the form $H = -\Delta + w + gv$, where $w(x)$ is a real-valued periodic function, g is a positive constant, and $v(x)$, $x \in \mathbb{Z}^d$, are independent, identically distributed real random variables. We prove that if the operator $-\Delta + w$ has gaps in the spectrum and g is sufficiently small, then the operator H develops pure point spectrum with exponentially decaying eigenfunctions in a vicinity of the gaps.

KEY WORDS: Random media; random patentials; Anderson model; Schrödinger operators; localization; gaps in the spectrum.

INTRODUCTION

We consider a matrix operator $H = H_0 + gv$ acting in $l^2(\mathbb{Z}^d)$ as follows:

$$(H\psi)(x) = \sum_{y \in \mathbb{Z}^d} H_0(x, y)\, \psi(y) + gv(x)\, \psi(x), \qquad x \in \mathbb{Z}^d \tag{1}$$

where $v(x)$, $x \in \mathbb{Z}^d$, are real, independent, identically distributed random variables, g is a positive constant, and H_0 is a local periodic operator in the following sense: there exists a natural number ρ (called the range of H_0) such that if $|x-y| > \rho$, then $H_0(x, y) = 0$, and there exists a vector $q = (q_1, ..., q_d) \in \mathbb{Z}^d$ with positive components such that $H_0(x, y) = H_0(x+q', y+q')$, $\forall x, y \in \mathbb{Z}^d$ and $\forall q' \in q_1 \mathbb{Z} \times \cdots \times q_d \mathbb{Z}$. We show that the spectrum of such an operator H_0 consists of a finite number of intervals which we shall call bands of the spectrum, the intervals between bands of

[1] Department of Mathematics, University of North Carolina, at Charlotte, Charlotte, North Carolina 28223. figotin@mosaic.uncc.edu.

[2] Department of Mathematics, University of California at Irvine, Irvine, California 92717-3875.

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the spectrum (if any) being the gaps in the spectrum. One can easily construct local periodic operators exhibiting gaps in the spectrum. For instance, let $H_0 = -\Delta + aw$, where Δ is the lattice Laplacian, a is a positive constant, and w is the operator of the multiplication by a real, periodic, nonconstant function $w(x)$, so H_0 is a local periodic operator. Since Δ is a bounded operator, it is clear that H_0 has gaps in the spectrum if the constant a is large enough. Another example of a periodic operator H_0 exhibiting gaps in the spectrum is constructed in ref. 1.

According to the philosophy of Anderson localization, localized states can appear in a vicinity of movable edges of gaps in the spectrum, i.e., such edges that depend on random coefficients.[2,3] It is known that operators of the form (1) with probability 1 have pure point spectrum with exponen tially decaying eigenfunctions for low energies, i.e., far enough from the spectrum of H_0,[4-11] and also near the endpoints of the spectrum.[15] We prove here that if the spectrum of the operator H_0 has gaps, then for a sufficiently small constant g the random operator H with probability 1 develops pure point spectrum with exponentially decaying eigenfunctions in a vicinity of all gaps of the operator H_0.

Our proof of localization in the gaps is based on the multiscale method used by von Dreifus and Klein[9] and Spencer[15] and on the relevant spectral properties of periodic operators and their restrictions to finite domains that we develop in this paper.

1. STATEMENT OF RESULTS

We begin with a precise definition of a local periodic operator. Let D be a natural number and $l^2(\mathbb{Z}^d, \mathbb{C}^D)$ be the Hilbert space of $\mathbb{C}^D$-valued functions $\varphi(x)$, with the standard norm $\|\varphi\|^2 = \sum |\varphi(x)|^2$. Let us denote by $\mathscr{L}_D$ the linear space of all $\mathbb{C}^D$-valued functions $\varphi(x)$. If $D = 1$, we shall just write $l^2(\mathbb{Z}^d)$ and $\mathscr{L}$ in place of $l^2(\mathbb{Z}^d, \mathbb{C}^1)$ and $\mathscr{L}_1$, respectively. Now we introduce a matrix H_0 with entries $H_0(x, y)$, x, $y \in \mathbb{Z}^d$, which are in turn $D \times D$ matrices with complex entries. We shall consider here just symmetric matrices H_0; thus $H_0(x, y) = H_0^*(y, x)$, x, $y \in \mathbb{Z}^d$, where for a matrix (operator) A the adjoint to its matrix (operator) is denoted by A^*. We define a norm $|x|_\infty$ for $x = (x_1, ..., x_d) \in \mathbb{Z}^d$ as follows:

$$|x|_\infty = \max_{1 \leqslant j \leqslant d} |x_j|$$

Definition. We shall call a matrix A local if there is a natural number ρ such that $A(x, y) = 0$ whenever $|y - x|_\infty > \rho$. For a vector $q =$

$(q_1,..., q_d) \in \mathbb{Z}^d$ with positive coordinates we shall call a matrix A q-periodic (or just periodic) if it is local and the following equalities hold:

$$A(x, y) = A(x+q', y+q'), \qquad \forall x, y \in \mathbb{Z}^d, \quad \forall q' \in q_1\mathbb{Z} \times \cdots \times q_d\mathbb{Z} \quad (1.1)$$

We associate with any periodic matrix H_0 and operator denoted by same symbol whose action is defined in standard fashion by $(H_0\psi)(x) = \sum_y H_0(x, y)\, \psi(y)$. Clearly, a periodic operator H_0 is correctly defined as an operator from $\mathscr{L}_D$ to $\mathscr{L}_D$ and it is a bounded self-adjoint operator in $l^2(\mathbb{Z}^d, \mathbb{C}^D)$. In particular, a q-periodic operator H_0 maps any q-periodic function ψ onto a q-periodic function $H_0\psi$.

Remark. If $H_0 = -\Delta + w$, where w is the operator of the multiplication by a q-periodic function, then H_0 is a q-periodic operator.

Schrödinger operators with periodic potentials on $\mathbb{R}^d$ are the subject of the well-known Floquet–Bloch theory.[12] Since modifications needed to extend the theory to the lattice case are hard to find in the literature, we will state and prove what we need.

Theorem 1 (Band structure of spectrum). If H_0 is a periodic operator on $l^2(\mathbb{Z}^d, \mathbb{C}^D)$, then its spectrum σ_0 consists of a finite number J of intervals, namely

$$\sigma_0 = \bigcup_{1 \leqslant i \leqslant J} [\mu_i^{(0)}, \lambda_i^{(0)}]; \qquad 0 \leqslant \mu_i^{(0)} \leqslant \lambda_i^{(0)}, \qquad 1 \leqslant i \leqslant J;$$
$$\lambda_i^{(0)} < \mu_{i+1}^{(0)}, \quad 1 \leqslant i \leqslant J-1 \qquad (1.2)$$

Definition (Gaps). We call the above intervals bands. If $J > 1$, then we shall call the intervals $(\lambda_i^{(0)}, \mu_{i+1}^{(0)})$, $1 \leqslant i \leqslant J-1$, gaps in the spectrum (or just gaps).

We have already discussed in the introduction that periodic operator with gaps in the spectrum can be easily constructed; in particular, the lattice Schrödinger operator of the form $H_0 = -\Delta + w$ with a periodic potential may have gaps in the spectrum. Thus, we shall just assume the existence of gaps in the spectrum of the operator H_0.

From now on we always have $D = 1$, unless stated otherwise. The main operator we are interested in is the operator $H = H_0 + gv$, where g is a positive constant and the operators H_0 and v satisfy the following assumptions:

Assumption H. H_0 is a q-periodic self-adjoint operator on $l^2(\mathbb{Z}^d)$ with $J-1 > 0$ gaps $(\lambda_i^{(0)}, \mu_{i+1}^{(0)})$, $1 \leqslant i \leqslant J-1$.

Assumption V. v is the operator on $l^2(\mathbb{Z}^d)$ given by multiplication by $v(x)$, where $v(x)$, $x \in \mathbb{Z}^d$, are independent, identically distributed random real-valued variables on a probability space with probability measure $\mathbb{P}$. The probability distribution μ of $v(0)$ has a bounded density φ with $\|\varphi\|_\infty \leqslant D_0$. For convenience we take $\mathscr{R}(v(x)) = [-1, 1]$, where $\mathscr{R}(v(0))$ is the essential range of the random variable $v(0)$.

Theorem 2 (Location of the spectrum). Let $\xi(x) = \xi_\omega(x)$, $x \in \mathbb{Z}^d$, be a set of real-valued, independent, identically distributed random variables on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ $(\omega \in \Omega)$ such that for some finite constant ξ_1, ξ_2 we have

$$\mathscr{R}(\xi(x)) = [\xi_1, \xi_2] \tag{1.3}$$

Suppose that the operator H acts in the Hilbert space $l_2(\mathbb{Z}^d)$ and $H = H_0 + \xi$, where H_0 satisfies Assumption H and ξ is the operator given by multiplication by the function $\xi(\cdot)$. Then the following statements hold:

(i) With probability 1 the spectrum $\sigma(H)$ of the operator H is nonrandom, i.e., there exists a closed set $\sigma \subseteq \mathbb{R}$ such that with probability 1, $\sigma(H) = \sigma$; in addition, with probability 1 the spectrum can be represented as follows:

$$\sigma(H) = \sigma = \sigma(H_0) + \mathscr{R}(\xi(x)) = \sigma(H_0) + [\xi_1, \xi_2] \tag{1.4}$$

where for two subsets $A, B \subseteq \mathbb{R}$, $A + B = \{\lambda + \mu \colon \lambda \in A, \mu \in B\}$.

(ii) Let us set $\xi(x) = gv(x)$, where v satisfies Asdsumption V; if we use the notations of TGheorem 1 and introduce g_i by the equality

$$g_i = (\mu_{i+1}^{(0)} - \lambda_i^{(0)})/2, \qquad 1 \leqslant i \leqslant J-1 \tag{1.5}$$

then for any $0 \leqslant g < g_i$ with probability 1 the spectrum $\sigma(H) = \sigma$ has a nonempty gap

$$(\lambda_i, \mu_{i+1}), \qquad \lambda_i = \lambda_i^{(0)} + g < \mu_{i+1} = \mu_{i+1}^{(0)} - g \tag{1.6}$$

which is associated naturally with the gap $(\lambda_i^{(0)}, \mu_{i+1}^{(0)})$ in the spectrum of the unperturbed periodic operator.

In other words, Theorem 2 says that the spectrum of the random operator H is nonrandom and if the constant g is smal enough, then it has a band-gap structure associated naturally with the spectrum of the operator H_0. Moreover, taking the coefficient g small enough, we can open up any gap in the spectrum of the unperturbed periodic operator.

The main statement of this paper is the following.

Theorem 3. Let $H = H_0 + gv$, where v and H_0 satisfy Assumptions V and H, respectively. Assume also that for some i, $1 \leqslant i \leqslant J-1$, we have $0 \leqslant g < g_i$ [so (λ_i, μ_{i+1}) is a gap in the spectrum of H with probability 1]. Then for any Ω_+, $0 < \Omega_+ < 1$, there exists $\tilde{p}_+ = \tilde{p}_+(d, H_0, D_0, \Omega_+, g) > 0$, such that if the distribution μ of $v(0)$ satisfies the condition $p_+ \equiv \mu\{[\Omega_+, 1] < \tilde{p}_+$, the operator H is exponentially localized in the interval $(\lambda_i - \delta_+, \lambda_i)$, for some $\delta_+ > 0$, with probability 1. Moreover,

$$\lim_{p_+ \to 0} \delta_+ = g(1 - \Omega_+) \tag{1.7}$$

Similarly, given $-1 < \Omega_- < 0$, there exists $\tilde{p}_- = \tilde{p}_-(d, H_0, D_0, \Omega_-, g) > 0$ such that if $p_- \equiv \mu\{[-1, \Omega_-]\} < \tilde{p}_-$, H is exponentially localized in the interval $(\mu_i, \mu_i + \delta_-)$ for some $\delta_- > 0$ with probability 1, with a similar statement to (1.7) for δ_-.

We also prove a somewhat different version of Theorem 3.

Theorem 3′. Let $H = H_0 + gv$ as in Theorem 3, and in addition suppose that $\mu\{|v(0) \pm 1| \leqslant \varepsilon\} \leqslant C\varepsilon^\eta$ for a finite constant C and a constant $\eta > d$. Then, if $0 \leqslant g \leqslant g_i$, we can find $\delta_\pm(d, H_0, D_0, g, C, \eta)$ such that H is exponentially localized in the intervals $(\lambda_i - \delta_+, \lambda_i)$, $(\mu_i, \mu_i + \delta_-)$ with probability 1.

The proofs of Theorems 2, 3, and 3′ are based on auxiliary statements concerning the relationship of the spectrum of a periodic operator A and its periodic restrictions to finite parallelepides in $\mathbb{Z}^d$: they will be formulated as theorems below. In order to so, we introduce the following notations. If $u, v \in \mathbb{Z}^d$, then $uv = (u_1 v_1, ..., u_d v_d) \in \mathbb{Z}^d$.

Definition. Let $u, v \in \mathbb{N}^d$. If $v = nu$ for some $n \in \mathbb{N}^d$ we will write $u \preccurlyeq v$. If in addition all the coordinates of n are strictly greater than 1, we will write $u \prec v$.

Definition. For $u \in \mathbb{N}^d$ we define a parallelepiped

$$C^u = \{0, ..., u_1 - 1\} \times \cdots \times \{0, ..., u_d - 1\} \subset \mathbb{Z}^d$$

We will write $C_u \preccurlyeq C_v$ or $C_u \prec C_v$ if $u \preccurlyeq v$ or $u \prec v$, respectively.

Suppose now that A is a q-periodic self-adjoint operator in $l^2(\mathbb{Z}^d, \mathbb{C}^D)$ and $u \succcurlyeq q$. Then we introduce a finite matrix $\mathring{A}_{C^u}$ associated with the operator A as follows. Let

$$\mathring{A}_{C^u}(x, y) = \sum_{n \in \mathbb{Z}^d} A(x, y + nu), \qquad x, y \in \mathbb{Z}^d \tag{1.8}$$

Now, we define

$$\mathring{A}_{C^u} = \{\mathring{A}_{C^u}(x, y), x, y \in C^u\}$$

If $u = q$, will shall just write

$$\mathring{A} = \mathring{A}_{C_q} \tag{1.9}$$

We call the matrix $\mathring{A}_{C^u}$ the periodic restriction of the local operator A to the parallelepiped C^u, $u \succcurlyeq q$. Let us denote by $\sigma(A)$ the spectrum of an operator (or matrix) A.

Theorem 4. Let A be a q-periodic self-adjoint operator in $l^2(\mathbb{Z}^d, \mathbb{C}^D)$. Suppose that C_n, $n = 1, 2,...$, is a sequence of parallelepipeds such that $C^q \preccurlyeq C_n \prec C_{n+1}$, $n \geqslant 1$. Then

$$\sigma(A) = \overline{\bigcup_{n \geqslant 1} \sigma(\mathring{A}_{C_n})}, \qquad \sigma(\mathring{A}_{C_n}) \subseteq \sigma(\mathring{A}_{C_{n+1}}) \subseteq \sigma(A) \tag{1.10}$$

This theorem enables us to control ther spectrum in vicinities of gaps of the periodic restrictions of the operator H to finite parallelepipeds.

2. PROOF OF THEOREMS 1, 2, AND 4

In this section we investigate the location of the spectrum of the operators H and H_0. We need first to extend some aspects of the well-known Floquet–Bloch theory to the periodic operatoprs H_0 following the scheme developed for multidimensional periodic Schrödinger operators in ref. 12.

Floquet–Bloch Theory for Lattice Periodic Operators

Let A be a q-periodic self-adjoint operator in $\mathscr{L}_D$ with entries $A(x, y)$, $x, y \in \mathbb{Z}^d$, defined in the previous section, and let V_j, $1 \leqslant j \leqslant d$, be the unitary shift operators acting on Hilbert spaces $l^2(\mathbb{Z}^d, \mathbb{C}^D)$ which acts as follows. If e_j, $1 \leqslant j \leqslant d$, are the standard basis vectors in the lattice $\mathbb{Z}^d$, then V_j are defined by formulas

$$(V_j \Psi)(x) = \Psi(S_j(x)), \qquad S_j(x)), \qquad S_j(x) = x - e_j, \qquad x \in \mathbb{Z}^d, \quad 1 \leqslant j \leqslant d \tag{2.1}$$

That is, S_j stands for the shift in the lattice $\mathbb{Z}^d$ by the vector e_j. To proceed further we need an appropriate description of q-periodic operators. We adopt here the following notations:

M^D is the set of $D \times D$ matrices with complex entries.

$\mathscr{F}^D_{qa}$ is the set of q-periodic $\mathbb{C}^D$-valued function $\Psi(x)$, $x \in \mathbb{Z}^d$.

$\mathscr{M}^D_q$ is the set of q-periodic M^D-valued functions $a(x)$, $x \in \mathbb{Z}^d$.

$\mathscr{A}^D_q$ is the set of q-periodic operators.

$V^z = V_1^{z_1} \cdots V_d^{z_d}$, $z \in \mathbb{Z}^d$.

If $a(\cdot) \in \mathscr{F}^D_q$ and $z \in \mathbb{Z}^d$, $a^{(z)}(x) = a(x - z)$, $x \in \mathbb{Z}^d$.

Lemma 2.1. Let a be the operator given by multiplication by the periodic function $a(\cdot) \in \mathscr{M}^D_q$. Then:

(i) For any $a(\cdot) \in \mathscr{M}^D_q$ and $z \in \mathbb{Z}^d$, a, $V^z \in \mathscr{A}^D_q$.

(ii) A is a periodic operator with entries $A(x, y)$, x, $y \in \mathbb{Z}^d$, i.e., $A \in \mathscr{A}^D_q$ if and only if there exist a finite positive ρ and a collection of q-periodic functions $a_z(\cdot) \in \mathscr{F}^D_q$, $z \in \mathbb{Z}^d$, and $|z| \leqslant \rho$ such that the following representation is true:

$$A = \sum_{|z| \leqslant \rho} a_z V^z, \qquad a_z(x) = A(x, x - z), \qquad x \in \mathbb{Z}^d \tag{2.2}$$

If in addition A is a self-adjoint operator, then the following equalities hold:

$$a_z^*(x) = a_{-z}(x - z) = a_{-z}^{(z)}(x), \qquad x, z \in \mathbb{Z}^d, \qquad |z| \leqslant \rho \tag{2.3}$$

Moreover, $\mathscr{A}^D_q$ is an algebra and for any $a(\cdot) \in \mathscr{F}^D_q$ and $z \in \mathbb{Z}^d$ we have

$$aV^z = V^z a^{(-z)} \tag{2.4}$$

Proof. The proof follows immediately from the definition of a q-periodic operator and operators V_j. ∎

For any parallelepiped C^u, $u \succcurlyeq q$, and a q-periodic operator A we have defined the matrix $\mathring{A}_{C^u}$ by formula (1.8) and called it the periodic restriction of A to C^u. This periodic restriction possesses the following properties.

Lemma 2.2. Let A be a q-periodic operator with entries $A(x, y)$, $x, y \in \mathbb{Z}^d$, and C^u, $u \succcurlyeq q$. Then the function $\mathring{A}_{C^u}(x, y)$ defined by formula (1.8) for any $x, y \in \mathbb{Z}^d$ is u-periodic with respect to both x and y. Namely

$$\mathring{A}_{C^u}(x + nu, y) = \mathring{A}_{C^u}(x, y + nu) = \mathring{A}_{C^u}(x, y), \qquad x, y, n \in \mathbb{Z}^d \tag{2.5}$$

In addition, if A is a self-adjoint operator, then the finite matrix $\mathring{A}_{C^u}(x, y)$, $x, y \in C^u$, is also self-adjoint. If B is another q-periodic operaor, then the following identity holds:

$$(\mathring{AB})_{C^u} = \mathring{A}_{C^u} \mathring{B}_{C^u} \tag{2.6}$$

Proof. The statements of the lemma easily follow from the definition of q-periodic operators, in particular (1.1). ∎

It is clear from (2.4) that a q-periodic A commutes with the operators $V_j^{q_j}$, $1 \leqslant j \leqslant d$. Based on this fact, we shall introduce an operator $\hat{A}$ which is on one hand unitarily equivalent to A, and on the other hand can be decomposed into fibers $\hat{A}(\kappa)$ by the direct integral

$$\hat{A} = \int_M^{\oplus} \hat{A}(\kappa)\, d\kappa, \qquad M = [0, q_1^{-1}] \times \cdots \times [0, q_d^{-1}] \tag{2.7}$$

where $\hat{A}(\kappa)$ is a $|Q| \times |Q|$ matrix depending on κ. In order to do so, we consider the Fourier transform F for $\Psi \in l^2(\mathbb{Z}^d, \mathbb{C}^D)$ defined by the formulas

$$[F\Psi](k) = \tilde{\Psi}(k) = \sum_{x \in \mathbb{Z}^d} e^{2\pi i k.x}\, \Psi(x) \tag{2.8}$$

$$\Psi(x) = [F^{-1}\tilde{\Psi}](x) = \int_K \tilde{\Psi}(k)\, e^{-2\pi i k.x}\, dk, \qquad K = [0, 1]^d \tag{2.9}$$

which is a unitary transform of $l^2(\mathbb{Z}^d, \mathbb{C}^D)$ to $L^2(K, \mathbb{C}^D)$, i.e., the Hilbert space of $\mathbb{C}^D$-valued functions on K which are square-integrable with respect to Lebesgue measure dk. We shall also consider the Fourier transform of the operator A and denote it by $\tilde{A} = FAF^{-1}$. It follows from the previous formula that $\tilde{\Psi}(k)$ can be viewed as a $(1,..., 1)$-periodic function on $\mathbb{R}^d$.

Now, to use the q-periodicity of the operator A and to handle q-periodic functions on the lattice $\mathbb{Z}^d$ it is convenient to introduce the discrete torus

$$\mathfrak{Q} = \mathfrak{Q}_q = \mathbb{Z}^d / \mathbb{Z}_q^d, \qquad \mathbb{Z}_q^d = q_1 \mathbb{Z} \times \cdots \times q_d \mathbb{Z} \tag{2.10}$$

where $\mathbb{Z}^d$ is treated as a ring with the ordinary operation of addition and the following operation of multyiplication for $a, b \in \mathbb{Z}^d$: $(ab)_j = a_j b_j$, $1 \leqslant \; \leqslant j \leqslant d$. Clearly, $\mathfrak{Q}$ as a set can be identified naturally with the parallelepiped $Q = C^q$, and we will identify a q-periodic complex-valued function on $\mathbb{Z}^d$ with the appropriate function on $\mathfrak{Q}$ (or Q). The space of $\mathbb{C}^D$-valued functions on $\mathfrak{Q}$ will be denoted by $\mathbb{C}^{D,\mathfrak{Q}}$, We introduce the scalar product for Φ, $\Psi \in \mathbb{C}^{\mathfrak{Q}}$ by

$$\Phi \cdot \Psi = \sum_{m \in \mathfrak{Q}} \Phi_m^* \Psi_m \tag{2.11}$$

where Φ^* is the vector adjoint to Φ. We also introduce the Fourier transform $\tilde{\Psi} = F_q \Psi$ of the $\mathbb{C}^D$-valued functions Ψ on the discrete torus $\mathfrak{Q}$ in the ordinary way by

$$\tilde{\Psi}_l = [F_q \Psi]_l = |Q|^{-1/2} \sum_{m \in \mathfrak{Q}} e^{2\pi i m l/q} \Psi_m, \qquad l \in \mathfrak{Q}, \quad F_q^* F_q = I \quad (2.12)$$

where I stands for the identity matrix and F_q^* is the matrix adjoint to F_q. In fact, F_q is a unitary matrix.

Returning to the construction of the direct integral (2.7), we decompose the parallelepiped K into equal small parallelepipeds as follows:

$$K = \bigcup_{l \in Q} M_l, \qquad M_l = M + l/q, \qquad l = (l_1, ..., l_d), \qquad q = (q_1, ..., q_d) \in \mathbb{Z}^d \quad (2.13)$$

where

$$l/q = (l_1/q_1, ..., l_d/q_d)$$

and consider the corresponding decomposition of a function $\tilde{\Psi} \in L^2(K, \mathbb{C}^D)$,

$$\tilde{\Psi}: \quad \{\tilde{\Psi}_l(\kappa), \kappa \in M, l \in Q\}, \qquad \tilde{\Psi}_l(\kappa) = \tilde{\Psi}(\kappa + l/q) \quad (2.14)$$

It follows from this formula that the function $\tilde{\Psi}_l(\kappa)$ is a q-periodic function of $l \in \mathbb{Z}^d$. So, if we introduce $\hat{\Psi}(\kappa) = \{\tilde{\Psi}_l(\kappa), l \in Q\}$ and the Hilbert space $L^2(M, \mathbb{C}^{D, \mathfrak{Q}})$ (i.e., the Hilbert space of $\mathbb{C}^{D, \mathfrak{Q}}$-valued functions on M which are square-integrable with respect to Lebesgue measure $d\kappa$), then based on the formula (2.14) one can define the unitary operator W,

$$[W\tilde{\Psi}](k) = \hat{\Psi}(\kappa), \qquad W: \quad L^2(K, \mathbb{C}^D) \mapsto L^2(M, \mathbb{C}^{D, \mathfrak{Q}}) \quad (2.15)$$

Therefore, we have the following representation of $L^2(K, \mathbb{C}^D)$ by the constant fiber direct integral:

$$WL^2(K, \mathbb{C}^D) = L^2(M, \mathbb{C}^{D, \mathfrak{Q}}) = \int_M^{\oplus} \mathbb{C}^{D, \mathfrak{Q}} \, d\kappa \quad (2.16)$$

For an operator A in $l^2(\mathbb{Z}^d, \mathbb{C}^D)$ we shall denote $\hat{A} = WFA(WF)^{-1}$. From the definitions (2.1) of the operators V_j we easily obtain

$$[\hat{V}_j \hat{\Psi}]_l(\kappa) = \exp\{2\pi i(\kappa_j + l_j/q_j)\} \; \hat{\Psi}_l(\kappa), \qquad 1 \leqslant j \leqslant d, \quad l \in \mathfrak{Q}, \quad \kappa \in M \quad (2.17)$$

In order to find the appropriate representation for the operator A, we use Lemma 2.1 and represent q-periodic functions $a_z(x)$ as follows:

$$a_z(x) = \sum_{l \in Q} \check{a}_{z,l} \exp\{-2\pi i(l/q)x\}, \qquad \check{a}_{z,l+\alpha q} = \check{a}_{z,l}, \qquad x, l, \alpha \in \mathbb{Z}^d \tag{2.18}$$

where

$$\check{a}_{z,l} = |Q|^{1/2} [F_q a'_z]_l, \qquad a'_z = [a_{z,m}, m \in \Omega\}, \qquad a_{z,m} = a_z(m), \qquad m \in \Omega \tag{2.19}$$

Then, taking into account (2.14), we get

$$\begin{aligned} [a_z \widetilde{\Psi}](k) &= \sum_{m \in Q} \check{a}_{z,m} \tilde{\Psi}(k - m/q) \\ [\hat{a}_z \hat{\Psi}]_l(\kappa) &= \sum_{m \in \Omega} \check{a}_{z,l-m} \hat{\Psi}_m(\kappa), \qquad l \in \Omega \end{aligned} \tag{2.20}$$

For any operator (matrix) B acting in the finite-dimensional space $\mathbb{C}^{D,\Omega}$ we shall denote by $\check{B}$ the following operator (matrix):

$$\check{B} = F_q B F_q^{-1} \tag{2.21}$$

Lemma 2.3. Let U_j, $1 \leqslant j \leqslant d$, be the unitary matrices on $\mathbb{C}^{D,\Omega}$ defined by

$$[U_j \Psi\}]_l = \Psi_{l-e_j}, \qquad l \in \Omega \tag{2.22}$$

and hence

$$[\check{U}_j \Psi]_l = \exp\{2\pi i l_j/q_j\} \, \Psi_l, \qquad l \in \Omega \tag{2.23}$$

Let b_l be an $\mathcal{M}^D$-valued function on the torus Ω and denote by b the operator given by multiplication by the function b_l in the finite-dimensional space $\mathbb{C}^{D,\Omega}$. Write $b_l^{(z)} = b_{l-z}$, $l \in \Omega$, $z \in \mathbb{Z}^d$, where $l - z$ is understood modulo q. Then the following relationships hold:

$$[\check{b}\Psi]_l = \sum_{m \in \Omega} \check{b}_{l-m} \Psi_m, \qquad l \in \Omega \tag{2.24}$$

$$b U^z = U^z b^{(-z)}, \qquad z \in \mathbb{Z}^d \tag{2.25}$$

Proof. The statement of the lemma follows immediately from (2.12) and (2.21). ∎

Lemma 2.4. Let $\check{a}_z$, $|z| \leqslant \rho$, be matrices on $\mathbb{C}^{D,\mathfrak{Q}}$ defined by

$$[\check{a}_z \Psi]_l = \sum_{m \in \mathfrak{Q}} \check{a}_{z,l-m} \Psi_m, \qquad l \in \mathfrak{Q} \tag{2.26}$$

Then the following relationships are true:

$$\begin{aligned} &[\hat{V}_j \hat{\Psi}](\kappa) = e^{2\pi i \kappa_j} \check{U}_j \hat{\Psi}(\kappa) \\ &[\hat{A}\hat{\Psi}](\kappa) = \left[\sum_{|z| \leqslant \rho} \check{a}_z e^{2\pi i (\kappa z)} \check{U}^z \right] \hat{\Psi}(\kappa), \qquad \kappa \in M \end{aligned} \tag{2.27}$$

In addition, the operator A has the desired fiber structure (2.7) and for the matrices $\hat{A}(\kappa)$ the following representation is valid:

$$\hat{A}(\kappa) = \sum_{|z| \leqslant \rho} \check{a}_z e^{2\pi i (\kappa z)} \check{U}^z, \qquad \kappa \in M \tag{2.28}$$

The matrices $\hat{A}(\kappa)$, $k \in M$, are self-adjoint.

Proof. The proof of (2.27) follows straightforwardly from (2.2), (2.17), (2.19), and (2.20). In turn, the equality (2.28) is a consequence of (2.27) and (2.14)–(2.16). The self-adjointness of $\hat{A}(\kappa)$ follows from (2.28), (2.27), (2.3), and (2.25). ∎

Lemma 2.5. Let us introduce the following operators in $l^2(\mathbb{Z}^d, \mathbb{C}^D)$:

$$V_j(\kappa) = e^{2\pi i \kappa_j} V_j, \qquad 1 \leqslant j \leqslant d, \qquad A(\kappa) = \sum_{|z| \leqslant \rho} a_z V(\kappa)^z \tag{2.29}$$

Then,

$$F_q^{-1} e^{2\pi i \kappa_j} U_j F_q = \mathring{V}_j(\kappa), \qquad F_q^{-1} \hat{A}(\kappa) F_q = \mathring{A}(\kappa) \tag{2.30}$$

Proof. The statements of the lemma follows from (19) and Lemmas 2.2 and 2.4. ∎

Theorem 2.6. If $\mathcal{U} = F_q^{-1} W F$ and A is a q-periodic self-adjoint operator, then we have

$$\mathcal{U} A \mathcal{U}^{-1} = \int_M^{\oplus} \mathring{A}(\kappa)\, d\kappa, \qquad M = [0, q_1^{-1}] \times \cdots \times [0, q_d^{-1}] \tag{2.31}$$

where the direct integral decomposition acts in the Hilbert space

$$\int_M^{\oplus} \mathbb{C}^{D,\mathfrak{Q}}\, d\kappa \tag{2.32}$$

In particular, the spectrum $\sigma(A)$ can be represented in the form

$$\sigma(A) = \bigcup_{\kappa \in M} \sigma(\mathring{A}(\kappa)) \tag{2.33}$$

Proof. The quality (2.31) follows immediately from Lemmas 2.4 and 2.5, whereas the representation (2.33) is a direct consequence of (2.31). ▮

Proof of Theorem 1. In view of the representation (2.33), the spectrum $\sigma(A)$ is equal to the union of the range of values of the set of real functions $\lambda_l(\kappa)$, $\kappa \in M (l \in \mathfrak{Q})$, which are respectively the eigenvalues of the matrices $\mathring{A}(\kappa)$. It easily follows from Lemmas 2.4 and 2.5 that the matrices $\mathring{A}(\kappa)$, and therefore their eigenvalues, are continuous functions of κ. This means that the union of the sets described above must consist of a finite number of intervals. Thios completes the proof of Theorem 1. ▮

To prove Theorem 2 we will need some more auxiliary statements for the q-periodic operators. For a given parallelepiped C^u and a u-periodic $\mathbb{C}^D$-valued function $\Psi(x)$, $x \in \mathbb{Z}^d$, let us denote by $(\pi_{C^u} \Psi)(x)$, $x \in C^u$, its restriction to C^u. Clearly π_{C^u} is a one-to-one correspondence between u-periodic $\mathbb{C}^D$-valued functions on $\mathbb{Z}^d$ and all $\mathbb{C}^D$-valued functions on the parallelepiped C^u. The statement below is an immediate consequence of Lemma 2.2.

Corollary 2.7. Suppose that A is a q-periodic operator in $\mathscr{L}_D$, $C \succcurlyeq C^q$, and $\Psi_C(x)$, $x \in C$, is a $\mathbb{C}^D$-valued on C; then

$$\mathring{A}_C \Psi_C = \pi_C A \pi_C^{-1} \Psi_C \tag{2.34}$$

In addition, if $\Psi(x)$, $x \in \mathbb{Z}^d$, is a u-periodic $\mathbb{C}^D$-valued function and $C = C^u \succcurlyeq C^q$, then

$$A\Psi = \pi_C^{-1} \mathring{A}_C \pi_C \Psi \tag{2.35}$$

Lemma 2.8. Suppose that A is a q-periodic in $\mathscr{L}_D$ and $C^q \preccurlyeq C_1 \preccurlyeq C_2$. Then the following is true:

$$\sigma(\mathring{A}_{C_1}) \subseteq \sigma(\mathring{A}_{C_2}) \tag{2.36}$$

Moreover, the eigenfunctions of the matrix $\mathring{A}_{C_1}$ can be naturally extended to the corresponding eigenfunctions of the matrix $\mathring{A}_{C_2}$.

Proof. To prove the inclusion, suppose that λ is an eigenvalue of the matrix $\mathring{A}_{C_1}$. Then there is a function $\Psi_1(x)$, $x \in C_1$, such that

$$\mathring{A}_{C_1} \Psi_1(x) = \lambda \Psi_1(x), \qquad x \in C_1 \tag{2.37}$$

Now, let us extend the function $\Psi_1(x)$ periodically on C_2 as follows:

$$\Psi_2(x) = (\pi_{C_2} \pi_{C_1}^{-1} \Psi_1)(x), \qquad x \in C_2 \tag{2.38}$$

Then by a straightforward computation we obtain from (2.34) and ($\pi A \Psi$) the following:

$$\mathring{A}_{C_2} \Psi_2 = \pi_{C_2} A \pi_{C_1}^{-1} \Psi_1 = \pi_{C_2} \pi_{C_1}^{-1} \mathring{A}_{C_1} \Psi_1 = \lambda \Psi_2 \tag{2.39}$$

This means that $\lambda \in \sigma(\mathring{A}_{C_2})$, which completes the prove of the lemma. ∎

For the investigation of spectra we will need the following statement (e.g., ref. 13).

Lemma 2.9 (Distance to the spectrum). Let $\mathscr{H}$ be a separable Hilbert space and A be a self-adjoint operator in $\mathscr{H}$. Then if $\sigma(A)$ is the spectrum of A and λ is a real number, then

$$\operatorname{dist}\{\sigma(A), \lambda\} = \min_{\Psi \in \mathscr{H}, \|\Psi\| = 1} \|(A - \lambda) \Psi\| \tag{2.40}$$

Proof of Theorem 4. Let us prove first the inclusion in the formula (1.10). To do so, assume that for a real λ there exist a natural n such that λ is an eigenvalue of the matrix $\mathring{A}_{C_n}$, i.e., $\lambda \in \sigma(\mathring{A}_{C_n})$, and there is a vector $\Psi(x)$, $x \in C_n$, such that

$$\mathring{A}_{C_n} \Psi(x) = \lambda \Psi(x), \qquad x \in C_n \tag{2.41}$$

Now, from (2.34) we easily obtain

$$(A \pi_{C_n}^{-1} \Psi)(x) = \lambda (\pi_{C_n}^{-1} \Psi)(x), \qquad x \in \mathbb{Z}^d \tag{2.42}$$

which follows straightforwardly from the q-periodicity of the operator A as an operator in $\mathscr{L}_D$. Then for any $m > n$ we define

$$\Psi_m(x) = (\pi_{C_n}^{-1} \Psi)(x), \quad x \in C_m, \qquad \Psi_m(x) = 0, \quad x \notin C_m \tag{2.43}$$

Let us pick an arbitrary $\varepsilon > 0$ and introduce the following notation for a function $\Phi(x)$:

$$\|\Phi(x)\|_{C_m}^2 = \sum_{x \in C_m} |\Phi(x)|^2 \tag{2.44}$$

Let us introduce also, for each j, $1 \leqslant j \leqslant d$, the number r_j, which is the ratio of the corresponding edges of the parallelepipeds C_m and C_n. Then since the function $\pi_{C_n}^{-1} \Psi$ is periodic, it is easy to see that

$$\|(A - \lambda) \Psi_m\|_{C_m}^2 \leqslant C(|C_m|/|C_n|) \left(\sum_{1 \leqslant j \leqslant d} r_j^{-1} \right) \|\Psi\|_{C_n}^2 \tag{2.45}$$

where $C = C(\rho, \|A\|)$ is a constant dependent on the range ρ of localization for the operator A and its norm (see definition in Section 1). From the definition of Ψ_m it follows that

$$\|\Psi_m\|^2 = \|\Psi_m\|^2_{C_m} = (|C_m|/|C_n|)\ \|\Psi\|^2_{C_n} \tag{2.46}$$

Besides, from the definition of the sequence of parallelepipeds C_n it follows that for each j, $r_j \to \infty$ when $m \to \infty$. If we set now $\tilde{\Psi}_m(x) = \Psi_m(x)/\|\Psi_m(x)\|$, then from the relationships (2.45) and (2.46) and the previous comment we obtain for any given $\varepsilon > 0$ and for sufficiently large m

$$\|(A-\lambda)\ \tilde{\Psi}_m\| \leqslant \varepsilon \tag{2.47}$$

From this and Lemma 2.9 we obtain the desired inclusion in (1.10). Therefore we have

$$\sigma(A) \supseteq \overline{\bigcup_{n \geqslant 1} \sigma(\mathring{A}_{C_n})} \tag{2.48}$$

To complete the proof, we have to prove the inclusion opposite to the above. If we pick again a positive ε, then in view of Lemma 2.9 we can pick $\Psi \in l^2(\mathbb{Z}^d, \mathbb{C}^D)$ with norm 1 such that

$$\|(A-\lambda)\Psi\| \leqslant \varepsilon \tag{2.49}$$

Now we define for any m

$$\Psi_m(x) = \Psi(x), \quad x \in C_m, \qquad \Psi_m(x) = 0, \quad x \notin C_m \tag{2.50}$$

If $\tilde{\Psi}_m(x) = \Psi_m(x)/\|\Psi_m(x)\|$, then since the operator A has a bounded norm and vector Ψ belong to the corresponding Hilbert space and has norm 1, we can pick a sufficiently large m such that

$$\|(A-\lambda)\ \tilde{\Psi}_m\| \leqslant 2\varepsilon \tag{2.51}$$

Now we note that for any $n > m$, by (2.35), we have

$$\pi_{C_n}(A-\lambda)\ \tilde{\Psi}_m = (\mathring{A}_{C_n} - \lambda)\ \pi_{C_n} \tilde{\Psi}_m \tag{2.52}$$

In addition, the definition of Ψ_m yields

$$\|\pi_{C_n} \tilde{\Psi}_m\|_{C_n} = \|\tilde{\Psi}_m\| = 1 \tag{2.53}$$

From (2.52), (2.53), and (2.51) we conclude that

$$\|(\mathring{A}_{C_n} - \lambda)\ \pi_{C_n} \tilde{\Psi}_m\|_{C_n} \leqslant 2\varepsilon \tag{2.54}$$

Therefore for any ε there is an n such that

$$\operatorname{dist}\{\sigma(\mathring{A}_{C_n}), \lambda\} \leqslant 2\varepsilon \tag{2.55}$$

From this we may conclude that

$$\sigma(A) \subseteq \overline{\bigcup_{n \geqslant 1} \sigma(\mathring{A}_{C_n})} \tag{2.56}$$

The last relationship together with (2.48) implies the equality in (1.10) which together with Lemma 2.8 completes the proof of Theorem 4. ▮

Lemma 2.10. Suppose that the operator $A = B + \xi$ acts in $l_2(\mathbb{Z}^d)$, where B is a q-periodic self-adjoint operator and $\xi(x)$ is a u-periodic real-valued function such that $u \succcurlyeq q$ and for some finite constants ξ_1, ξ_2: $\xi_1 \leqslant \xi(x) \leqslant \xi_2$, $x \in \mathbb{Z}^d$. Then for any parallelepiped $C \succcurlyeq C^u$ the following is true:

$$\sigma(\mathring{A}_C) \subseteq \sigma(\mathring{B}_C) + [\xi_1, \xi_2] \subseteq \sigma(B) + [\xi_1, \xi_2] \tag{2.57}$$

$$\sigma(A) \subseteq \sigma(B) + [\xi_1, \xi_2] \tag{2.58}$$

Proof. Without loss of generality we may assume that $-\xi_1 = \xi_2 = \xi_0$, where ξ_0 is a nonnegative constant, since we can always redefine A as $A = (B + t) + (\xi - t)$, $t = (\xi_2 - \xi_1)/2$. Keeping this in mind, let us note now that for any two linear bounded operators D_1 and D_2

$$\sigma(D_1) \subseteq \sigma(D_2) + [-d, d], \qquad d = \|D_1 - D_2\| \tag{2.59}$$

Indeed, if $\lambda \notin \sigma(D_2) + [-d, d]$, then $\|(D_2 - \lambda)^{-1}\| < d^{-1}$ and therefore $(D_1 - \lambda)^{-1}$ is clearly a bounded operator, which implies (2.59). Since $\|\xi\| \leqslant \xi_0$, then (2.59) implies the first inclusion in (2.57) and (2.58). The second inclusion in (2.57) follows from the first one and (1.10). The lemma is therefore proved. ▮

Proof of Theorem 2. Let us note that without loss of generality we may assume that $u = (u_1, ..., u_d)$ and the parameter ρ associated with a u-periodic local operator A satisfy the following inequality:

$$\min_{1 \leqslant j \leqslant d} u_j > 2\rho + 1 \tag{2.60}$$

If not, we may always pick $u' \succ u$ such that u' satisfies (2.60) and treat A as u'-periodic. We shall assume from now on that the inequality (2.60) is sxatisfied for any period u we consider, in particular for $u = q$.

We have defined the periodic restriction $\mathring{A}_C$ for any q-periodic operator for $C = C^u$, $u \succcurlyeq q$. We need to extend properly this definition for local operators A which are not necessarily periodic. This can be done as follows. First of all, given a parallelepiped $C = C^u + l$, we construct an appropriate u-periodic operator associated with C and A, which we shall denote by $A^{(C)}$. We note that for a local operator A the representation (2.2) is clearly still valid. We want to preserve the self-adjointness for $A^{(C)}$ is A self-adjoint. The operator A is self-adjoint if and only if the constraints (2.3) hold. In order to provide these constraints, we represent the set $\{z \in \mathbb{Z}^d\colon |z| \leqslant \rho\} = \{0\} \cup Z \cup (-Z)$ in such a way that $0 \notin Z \cup (-Z)$ and $Z \cap (-Z) = \varnothing$. Clearly we can always do this. Then we may set a_z, $z \in Z \cup \{0\}$, as we wish and define a_z, $z \in (-Z)$, by the equalities (2.3). Now we define a linear operator τ_C which maps any $\mathbb{C}^D$-valued function $a(x)$, $x \in \mathbb{Z}^d$, onto a u-periodic function $\tau_C a$ as follows

$$a_z^{(C)}(x) = \tau_C a(x) = a(x), \quad x \in C, \qquad a_z^{(C)}(x + un) = a_z^{(C)}(x), \quad x \in \mathbb{Z}^d \tag{2.61}$$

In other words, $\tau_C a$ is a u-periodic extension of a coinciding with the function a on the parallelepiped $C = C^u + l$. Now since A is represented by (2.2), we define an associated u-periodic operator $A^{(C)}$ by the same formula (2.2) where the a_z, $z \in Z \cup \{0\}$, are replaced by $a_z^{(C)}$, $z \in U \cup \{0\}$, and the remaining functions $a_z^{(C)}$, $z \in (-Z)$, are defined to keep the constraints (2.3). With this definition the u-periodic operator $A^{(C)}$ associated with the self-adjoint operator A and the parallelepiped $C = C^u + l$ is also self-adjoint. Having this, we define the periodic restriction $\mathring{A}_C$ of a local operatior A on a parallelepiped $C = C^u + l$ using (1.8) as follows:

$$\mathring{A}_C = [\mathring{A}^{(C)}]_{C^u}, \qquad C = C^u + l \tag{2.62}$$

Definition. We say that a point x is a boundary point of a parallelepiped C if there exists j, $1 \leqslant j \leqslant d$, such that either $x + e_j \notin C$ or $x - e_j \notin C$. The set of boundary points is denoted by ∂C.

The statement below shows that the periodic restriction of A on C does not differ much from the regular restriction $A(x, y)$, $x, y \in \mathbb{Z}^d$.

Lemma 2.11. Let A be a local operator. If $C = C^u + l$, $l \in \mathbb{Z}^d$, then the following equalities are true:

$$\mathring{A}_C(x, y) = A(x, y), \qquad x, y \in C, \qquad \operatorname{dist}\{x, \partial C\}, \operatorname{dist}\{y, \partial C\} > \rho \tag{2.63}$$

where $\operatorname{dist}\{x, \partial C\} = \max_{z \in \partial C} |x - z|_\infty$. If A is a self-adjoint operator, then $\mathring{A}_C$ is self-adjoint as well.

Proof. The statements of the lemma follow straightforwardly from (1.8), (2.61), (2.62), and (2.60). ∎

The construction of the periodic restrictions is clearly applicable to the operators $H = H_0 + gv$ defined by (1). Whenever we shall need to emphasize that H depends on v we write $H = H(v)$.

Lemma 2.12. The spectrum of the operator H is nonrandom with probability 1, i.e., there exists a closed set $\sigma \subseteq \mathbb{R}$ such that with probability 1, $\sigma(H) = \sigma$.

Proof. We note that the operator H is metrically transitive and then we can just refer to ref. 14. ∎

Let $\mathscr{P}_q$ be the set of real-valued functions $\xi(x)$ which are u-periodic for some $u \succcurlyeq q$ and satisfy $\xi_1 \leqslant \xi(x) \leqslant \xi_2$.

Theorem 2.13. Suppose that C_n, $n = 1, 2, \ldots$, is a sequence of parallelepipeds such that $C^q \preccurlyeq C_n \prec C_{n+1}$, $n \geqslant 1$. Let the operator $H = H_0 + \xi$ and the spectrum σ be defined as in Theorem 2. Then the nonrandom spectrum σ of the operator H can be represented as follows:

$$\sigma = \overline{\bigcup_{\xi \in \mathscr{P}_q} \sigma[H(\xi)]} = \overline{\bigcup_{n \geqslant 1, \xi \in \mathscr{P}_q} \sigma[\mathring{H}_{C_n}(\xi)]} = \sigma(\xi_1, \xi_2) \tag{2.64}$$

where

$$\sigma(\xi_1, \xi_2) = \sigma(H_0) + [\xi_1, \xi_2] \tag{2.65}$$

Proof. First of all we note that the following equalities are true:

$$\overline{\bigcup_{\xi \in \mathscr{P}_q} \sigma[H(\xi)]} = \overline{\bigcup_{n \geqslant 1, \xi \in \mathscr{P}_q} \sigma[\mathring{H}_{C_n}(\xi)]} = \sigma(H_0) + [\xi_1, \xi_2] \tag{2.66}$$

These inequalities follow straightforwardly from Theorem 4 and Lemma 2.10 if we note that for a u-periodic ξ from $\mathscr{P}_q$ the operator $H(\xi)$ is u-periodic and, in addition, we may set $\xi(x) \equiv t$, where t is a constant such that $-1 \leqslant t \leqslant 1$.

Recall now that the function $\xi(x)$ is a random function, i.e., we have a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $\xi(x) = \xi_\omega(x)$, where ω is a realization from Ω. Let us observe that it follows from Lemma 2.12 that there exists a set $\Omega_1 \subseteq \Omega$ such that $\mathbb{P}(\Omega_1) = 1$ and

$$\sigma(H(\xi_\omega)) = \sigma, \; \omega \in \Omega_1 \tag{2.67}$$

Let us pick any positive ε and ω such that (2.67) is true. Assume that $\lambda \in \sigma$. Then in view of Lemma 2.9 there exists m and a vector Ψ in the Hilbert space such that $\|\Psi\| = 1$ and

$$\|(H(\xi_\omega) - \lambda)\Psi\| \leqslant \varepsilon, \qquad \Psi(x) = 0, \qquad x \notin C_m \tag{2.68}$$

We may impose the extra constraint $\Psi(x)=0$, $x \notin C_m$, on the vector Ψ since the operator H is local and bounded. Then for any $n > m$

$$H(\xi_\omega)\, \Psi(x) = \mathring{H}_{C_n}(\xi_\omega)\, \Psi(x), \qquad x \in C_n \tag{2.69}$$

and therefore

$$\|(\mathring{H}_{C_n}(\xi_\omega) - \lambda)\, \Psi\|_{C_n} \leqslant \varepsilon \tag{2.70}$$

The last equality implies that

$$\lambda \in \overline{\bigcup_{n \geqslant 1,\, \xi \in \mathscr{P}_q} \sigma[\mathring{H}_{C_n}(\xi)]}$$

and consequently

$$\sigma \subseteq \overline{\bigcup_{n \geqslant 1,\, \xi \in \mathscr{P}_q} \sigma[\mathring{H}_{C_n}(\xi)]} \tag{2.71}$$

To prove the opposite inclusion, let us pick again a positive ε and a u-periodic $\xi \in \mathscr{P}_q$. Then we suppose that $\lambda \in \sigma[H(\xi)]$. Since the operator H is local and bounded, we can apply again Lemma 2.9 and get for a natural m the equality (2.68) with ω dropped, i.e. there exists a vector Ψ, $\|\Psi\| = 1$, such that

$$\|(H(\xi) - \lambda)\, \Psi\| \leqslant \varepsilon, \qquad \Psi(x) = 0, \qquad x \notin C_m \tag{2.72}$$

Now we note that in view of the conditions imposed on $\xi_\omega(x)$ (see Theorem 2) for any positive δ there exist a set Ω_ξ, $\mathbb{P}(\Omega_\xi) = 1$, such that

$$\forall \delta, \forall \omega \in \Omega_\xi\colon \quad \exists l = l(\delta, \omega) \in \mathbb{Z}^d_u\colon \quad \max_{x \in C_m + l} |\xi_\omega(x) - \xi(x)| \leqslant \delta \tag{2.73}$$

Moreover, if we write $\Psi_l(x) = \Psi(x - l)$, then since ξ is u-periodic we have from (2.72)

$$\forall l \in \mathbb{Z}^d_u\colon \quad \|(H(\xi) - \lambda)\, \Psi_l\| \leqslant \varepsilon \tag{2.74}$$

Clearly, if we pick δ small enough, then

$$\forall \omega \in \Omega_\xi\colon \quad \exists l = l(\varepsilon, \omega) \in \mathbb{Z}^d_u\colon \quad \|(H(\xi_\omega) - \lambda)\, \Psi_l\| \leqslant 2\varepsilon \tag{2.75}$$

From this we immediately obtain

$$\sigma \supseteq \sigma[H(\xi)], \qquad \xi \in \mathscr{P}_q \tag{2.76}$$

and consequently

$$\sigma \supseteq \overline{\bigcup_{\xi \in \mathscr{P}_q} \sigma[H(\xi)]} \tag{2.77}$$

Thus, (2.66), (2.71), and (2.77) imply the desired relationship (2.64), which completes the prove of the theorem. ∎

In order to use the multiscale analysis[9] we need to get exponential estimates for the resolvent of the operators H and their periodic restrictions. For this purpose we will adapts the Combes–Thomas argument to our operators. We start with a description of the relevant resolvents. Let us denote by b_x, $x \in \mathbb{Z}^d$, the standard basis in the space $l^2(\mathbb{Z}^d)$, i.e., $b_x(x) = 1$, $b_x(y) = 0$, $y \neq x$, $y \in \mathbb{Z}^d$. In the case of $l^2(\mathbb{Z}^d, \mathbb{C}^D)$ we introduce the basis $b_{\alpha,x}$, $\alpha = 1,..., d$, i.e., $b_{\alpha,x}(\alpha, x) = 1$, and $b_{\alpha,x}(\beta, y) = 1$, if $\beta \neq \alpha$ or $y \neq x$, $\beta = 1,..., d$, $y \in \mathbb{Z}^d$. Suppose that A is a local operator (not necessarily periodic) acting in $l^2(\mathbb{Z}^d)$ or in $l^2(\mathbb{Z}^d, \mathbb{C}^D)$ with entries $A(x, y)$, $x, y \in \mathbb{Z}^d$. For such an operator the representation (2.2) is still applicable. Then if ζ is a complex or real number and $\zeta \notin \sigma(A)$, we may consider for the cases $l^2(\mathbb{Z}^d)$ or $l^2(\mathbb{Z}^d, \mathbb{C}^D)$, respectively, the Green's functions

$$G(\zeta, x, y) = (b_x, (H-\zeta)^{-1} b_y), \qquad x, y \in \mathbb{Z}^d \tag{2.78}$$

$$G(\zeta, x, y) = G(\zeta, \alpha, x, \beta, y) = (b_{\alpha,x}, (H-\zeta)^{-1} b_{\beta,y}), \qquad \alpha, \beta = 1,..., d, \qquad x, y \in \mathbb{Z}^d \tag{2.79}$$

We will often drop α and β in the notation of the resolvent for brevity.

Lemma 2.14. Suppose that A is a local operator described above such that for a positive constant c we have $|A(x, y)| \leqslant c$, $x, y \in \mathbb{Z}^d$. Suppose also that

$$\operatorname{dist}\{\zeta, \sigma(A)\} = \delta > 0 \tag{2.80}$$

Then there exists a positive constant $b = b(c, \rho)$ (ρ is the number associated with the local operator A) such that

$$|G(\zeta, x, y)| \leqslant 2\delta^{-1} e^{-b\delta|x-y|}, \qquad x, y \in \mathbb{Z}^d \tag{2.81}$$

where

$$|x| = \sum_{1 \leqslant j \leqslant d} |x_j| \tag{2.82}$$

Moreover, if A is a u-periodic operator, then the following identity is true:

$$G(\zeta, x+u, y+u) = G(\zeta, x, y), \qquad x, y \in \mathbb{Z}^d \tag{2.83}$$

Proof. For $\alpha \in \mathbb{C}^d$ let M_α be the operator given by multiplication by

$$M_\alpha(x) = e^{2\pi i(\alpha, x)}, \qquad x \in \mathbb{Z}^d \tag{2.84}$$

Then in view of (2.2) and (2.4) we have

$$A(\alpha)=M_\alpha A M_\alpha^{-1}=\sum_{|z|\leqslant\rho} a_z V(\alpha)^z, \qquad V_j(\alpha)=e^{2\pi i\alpha_j}V_j, \qquad 1\leqslant j\leqslant d \tag{2.85}$$

Note that $A(\alpha)$ coincides with the relevant operator in (2.29), but now $\alpha\in\mathbb{C}^d$. Clearly, the last representation implies the existence of a constant $K=K(c,\rho)$ such that

$$\|A-A(\alpha)\|\leqslant K|\alpha| \tag{2.86}$$

In view of (2.80) we have immediately $\|G(\zeta)\|\leqslant\delta^{-1}$. This inequality together with the inequality (2.86) implies for $G(\alpha,\zeta)=[A(\alpha)-\zeta]^{-1}$

$$\|G(\alpha,\zeta)\|\leqslant 2\delta^{-1}, \qquad |\alpha|<\delta/(2K) \tag{2.87}$$

Now we note that

$$[G(\alpha,\zeta)](x,y)=G(\zeta,x,y)\exp\{2\pi i\alpha(x-y)\}, x, y\in\mathbb{Z}^d \tag{2.88}$$

From this and the obvious inequality $|[G(\alpha,\zeta)](x,y)|\leqslant\|G(\alpha,\zeta)\|$ we obtain the inequality (2.81) by taking an appropriate α.

The identity (2.83) is a direct consequence of the u-periodicity of the operator A. This completes the proof of the lemma. ∎

Lemma 2.15. Suppose that the conditions of Lemma 2.14 are satisfied and let us consider for $C=C^u+l$, $l\in\mathbb{Z}^d$, the resolvent

$$G'_C(\zeta,x,y)=[(\mathring{A}_C-\zeta)^{-1}](x,y), \qquad x,y\in C \tag{2.89}$$

Then the following estimate is true:

$$|G'_C(\zeta,x,y)|\leqslant 2\delta^{-1}[1+2\Pi(v,\delta)]\,e^{-b\delta|x-u|_u}, \qquad x,y\in C \tag{2.90}$$

where b is the same constant as in Lemma 2.14 and

$$\Pi(v,\delta)=\prod_{1\leqslant j\leqslant d}(1-e^{-b\delta|u_j|})^{-1}, \qquad |x-y|_u=\min_{n\in\mathbb{Z}^d}|x-y-nu| \tag{2.91}$$

Proof. We note first that in view of the definition of the periodic restriction $\mathring{A}_C$ in (2.62) we may assume without loss of generality that A is a u-periodic operator and $C=C^u$. Keeping this in mind and using (2.83) together with the identity

$$\sum_{y\in\mathbb{Z}^d}[A(x,y)-\zeta]\,G(\zeta,y,z)=\delta_{x,z}, \qquad x,z\in\mathbb{Z}^d \tag{2.92}$$

where $\delta_{x,z}$ is the delta function, we obtain

$$\sum_{n\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d}[A(x,y)-\zeta]\,G(\zeta,y,z+un)=\sum_{n\in\mathbb{Z}^d}\delta_{x,z+un},\qquad x,z\in C \tag{2.93}$$

From this, (1.8), and (2.5) we obtain

$$\sum_{y\in C}[\mathring{A}_C(x,y)-\zeta]\,\mathring{G}_C(\zeta,y,z)=\delta_{x,z},\qquad x,z\in C \tag{2.94}$$

Therefore,

$$G'_C(\zeta,x,y)=\mathring{G}_C(\zeta,x,y)=\sum_{n\in\mathbb{Z}^d}G(\zeta,x,y+un),\qquad x,y\in C \tag{2.95}$$

From this and the previous lemma we immediately obtain

$$|G'_C(\zeta,x,y)|\leqslant 2\delta^{-1}\sum_{n\in\mathbb{Z}^d}e^{-b\delta|x-y-nu|},\qquad x,y\in C \tag{2.96}$$

If we recall the definition (2.91) of $|x-y|_u$ we can easily prove that there is $n'\in\mathbb{Z}^d$ such that

$$|x-y|_u=|z|,\qquad z=x-y-n'u=cu,\qquad 0\leqslant|c_j|\leqslant 1/2,\qquad 1\leqslant j\leqslant d \tag{2.97}$$

Now we rewrite the right side of the inequality (2.96) using (2.82) as follows:

$$\sum_{n\in\mathbb{Z}^d}e^{-b\delta|x-y-nu|}=\sum_{n\in\mathbb{Z}^d}e^{-b\delta|cu-nu|}=\prod_{1\leqslant j\leqslant d}\sum_{n\in\mathbb{Z}}e^{-b\delta|c_j-n|\cdot|u_j|} \tag{2.98}$$

We shall need the following elementary inequality:

$$\sum_{n\in\mathbb{Z}}e^{-c|m-n|}\leqslant e^{-c|m|}[1+2(1-e^{-c})^{-1}],\qquad 0\leqslant|m|\leqslant 1/2,\quad c>0 \tag{2.99}$$

which can be verified by a direct computation. Applying this inequality to the right side of (2.8) and combining the result with the inequality (2.96), we get the desired estimate (2.90). The lemma is proved. ∎

Proof of Theorem 3. Let us consider the left edge λ_i of the gap (λ_i,μ_i); the right edge μ_i can be treated in a similar way. We will use the conditions for localization given in Theorem 2.1 of von Dreifus and Klein.[9] We start with some definitions. For $u\in\mathbb{Z}^d$ we define $H^{(u)}$ by

$$H_0^{(u)}(x,y)=H_0(x+u,y+u),\qquad x,y\in\mathbb{Z}^d \tag{2.100}$$

We then set

$$H^{(u)} = H_0^{(u)} + gv, \qquad G^{(u)}(\zeta) = (H_0^{(u)} - \zeta)^{-1} \tag{2.101}$$

Notice that $\sigma(H^{(u)}) = \sigma$ with probability 1. For $l \in N$, $x \in \mathbb{Z}^d$, we define $\tilde{l} = l(1,..., 1)$ and $\Lambda_l(x) = C^{\tilde{l}} - [l/2] + x$ ($[y]$ is the entire part of the real number y) and for $\Lambda \subset \mathbb{Z}^d$

$$\partial_\rho \Lambda = \{ y \in \Lambda: \exists z \in \mathbb{Z}^d - \Lambda, |z - y|_\infty \leqslant \rho \} \tag{2.102}$$

Recall that ρ is the range of H_0. Also for $\Lambda \in \mathbb{Z}^d$ we write $H_\Lambda = \{ H(x, y), x, y \in \Lambda \}$, which is the matrix associated with the restriction of H to Λ with Dirichlet boundary conditions.

Definition 2.16. Let $x \in \mathbb{Z}^d$, $E \in \mathbb{R}$, $m > 0$, $l > \rho$. We say that $\Lambda_l(x)$ is (m, E)-regular if

$$\max_{u \in C^q} |G^{(u)}_{\Lambda_l(x)}(E; x, y)| \leqslant e^{-ml/2}, \qquad \forall y \in \partial_\rho \Lambda_l(x) \tag{2.103}$$

Otherwise we say that $\Lambda_l(x)$ is (m, E)-singular.

Let us fix $p > d$, an interval $I \subset \mathbb{R}$, m_0, and D_0 (see Assumption V). The von Dreifus–Klein criterion says that there exists $B = B(d, D_0, m_0, p) < \infty$ such that if

$$\mathbb{P}\{ \Lambda_{L_0}(x) \text{ is } (m_0, E)\text{-regular for all } E \in I \} \geqslant 1 - \frac{1}{L_0^p} \tag{2.104}$$

for some $L_0 > B$, then there exists $\delta = \delta(L_0, m_0, d, D_0, p) > 0$ such that the spectrum of H is exponentially localized in $(E_0 - \delta, E_0 + \delta)$.

Remark 2.17. Von Dreifus and Klein only discuss the case where $H = -\Delta + gv$. But their results are easily seen to extend to the case when $-\Delta$ is replaced by a translation-invariant operator with a finite range ρ. The remark that $-\Delta$ can be replaced by a q-periodic operator H_0 is due to Spencer,[16] who noticed that if the maximum over all translations of H_0 is introduced in the definition (2.103), the whole proof goes through.

Theorem 3 now follows from the following result.

Lemma 2.18. Let us fix $0 < \Omega_+ < 1$, and let $p_+ = \mu\{[\Omega_+, 1]\}$, $g_+ = g(1 - \Omega_+)$. If L is a sufficiently large positive integer such that $\tilde{L} \succcurlyeq q$, we have

$$\lim_{p_+ \to 0} \mathbb{P}\{ \Lambda_L(0) \text{ is } (b(g_+ - g')/4, \lambda)\text{-regular} \} = 1 \tag{2.105}$$

uniformly in $\lambda \in [\lambda_i - g', \lambda_i]$ for g', $0 < g' < g_+$, where b is given in Lemma 2.14.

Proof. Let $\mathscr{E}_L$ denote the event that $v(x) \leqslant \Omega_+$ for all $x \in \Lambda_L(0)$. If $\mathscr{E}_L$ occurs, and $0 < g' < g_+$, then for all $u \in C^q$ we have from (2.90) that for all $\lambda \in [\lambda_i - g', \lambda_i]$

$$|\mathring{G}^{(u)}_{\Lambda_L(0)}(\lambda; x, y)| \leqslant \frac{2^{d+1}}{g_+} \exp(-bg'' \, |x-y|_{\tilde{L}}) \tag{2.106}$$

for L sufficiently large in relation to q, for all $x, y \in \Lambda_L(0)$, where $g'' = g_+ - g'$. Define now $\Gamma^{(u)}_L$ by the equality

$$H^{(u)}_{0, \Lambda_L(0)} = \mathring{H}^{(u)}_{0, \Lambda_L(0)} + \Gamma^{(u)}_L \tag{2.107}$$

i.e., $\Gamma^{(u)}_L$ is the difference between matrices corresponding to the periodic and Dirichlet boundary conditions. Note that $\|\Gamma^{(u)}_L\| \leqslant C(H_0)$, where $C(H_0)$ is a constant which depends just on operator H_0. Then if G_Λ stands for the resolvent of the corresponding matrix H_Λ, the resolvent identity gives

$$G^{(u)}_{\Lambda_L(0)}(\lambda) = \mathring{G}^{(u)}_{\Lambda_L(0)}(\lambda) + \mathring{G}^{(u)}_{\Lambda_L(0)}(\lambda) \, \Gamma^{(u)}_L \, G^{(u)}_{\Lambda_L(0)}(\lambda)$$

$$G^{(u)}_{\Lambda_L(0)}(\lambda; 0, y) = \mathring{G}^{(u)}_{\Lambda_L(0)}(\lambda; 0, y) + \sum_{s, t \in \Lambda_L(0)} \mathring{G}^{(u)}_{\Lambda_L(0)}(\lambda; 0, t) \, \Gamma^{(u)}_L(t, s) \, G^{(u)}_{\Lambda_L(0)}(\lambda; s, y) \tag{2.108}$$

If $y \in \partial_\rho \Lambda_L(0)$, then using (2.106), we get

$$|G^{(u)}_{\Lambda_L(0)}(\lambda; 0, y)| \leqslant \frac{2^{d+1}}{g''} e^{-bg''(L/2-\rho)} + (2L+1)^{2d} \, C(H_0) \, \|G^{(u)}_{\Lambda_L(0)}(\lambda)\| \, e^{-bg''(L/2-\rho)} \tag{2.109}$$

since $\Gamma^{(u)}_L(t, s) = 0$ unless $s, t \in \partial_\rho \Lambda_L(0)$. Now let $\mathscr{W}_L(\lambda)$ be the event $\|G^{(u)}_{\Lambda_L(0)}(\lambda)\| \leqslant L^{2d}$ for all $u \in C^q$. Then we get

$$\begin{aligned} |G^{(u)}_{\Lambda_L(0)}(\lambda; 0, y)| &\leqslant \frac{2^{d+1}}{g''} \exp\left\{-bg''\left(\frac{L}{2} - \rho\right)\right\} [1 + (2L+1)^{2d} \, C(H_0) \, L^{2d}] \\ &\leqslant \exp\left(-\frac{bg''L}{8}\right) \end{aligned} \tag{2.110}$$

for all $\lambda \in [\lambda_i - g', \lambda_i]$, if L is greater than a finite constant $L'(d, b, g'', H_0)$. Thus

$$\mathbb{P}\{\Lambda_L(0) \text{ is } (bg''/4, \lambda)\text{-singular}\} \leqslant \mathbb{P}\{\mathscr{E}^c_L\} + \mathbb{P}\{\mathscr{W}^c_L(\lambda)\} \tag{2.111}$$

On the other hand, for all $\lambda \in [\lambda_i - g', \lambda_i]$,

$$\mathbb{P}\{\mathscr{E}^c_L\} \leqslant L^d \mathbb{P}(v(0) > \Omega_+\} \leqslant p_+ L^d \tag{2.112}$$

and by Wegner's estimate

$$\mathbb{P}\{\mathscr{W}_L(\lambda)\} \leqslant \frac{2D_0}{g} |C^q| \frac{L^d}{L^{2d}} = \frac{2D_0}{g} |C^q| L^{-d} \tag{2.113}$$

This completes the proof of the lemma, and hence Theorem 3. ▮

Proof of Theorem 3′. We use the localization criterion given by Spencer.[15] The proof is similar to the proof of Theorem 3, so we will only point out the differences. Lemma 2.18 is replaced by the following.

Lemma 2.19. Let $m_L = 2(d+2) \log L/L$. Under the hypotheses of Theorem 3′ we have

$$\limsup_{L \to \infty} \mathbb{P}\{\Lambda_L(0) \text{ is } (m_L, \lambda_i)\text{-regular}\} = 1 \tag{2.114}$$

Proof. The lemma is proved in a similar way to Lemma 2.18, for scales such that $\tilde{L} \succcurlyeq q$. Here we define $\mathscr{E}_L$ to be the event that $v(x) \leqslant 1 - \delta_L$ for all $x \in \Lambda_L(0)$, where $\delta_L = (\log L)^2/L$. By our assumptions we have

$$\mathbb{P}\{\mathscr{E}^c_L\} \leqslant L^d \mathbb{P}\{v(0) > 1 - \delta_L\} \leqslant C L^d \delta^\eta_L = C L^d \frac{(\log L)^{2\eta}}{L^\eta} \to 0 \qquad \text{as} \quad L \to \infty \tag{2.115}$$

since $\eta > d$.

Theorem 3′ now follows from Theorem 1 in ref. 15.

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